Alexander Duality; Some Applications in Combinatorics

Afshin Goodarzi
Freie Universität Berlin

The aim of these three lectures is to introduce Alexander duality and present some of its applications in combinatorics. Alexander duality can be seen as a vast generalisation of the Jordan Curve Theorem. It relates the connectivity (in the homological sense) of a space embedded in a sphere to the connectivity of the complement of the space.

Alexander duality has many applications in combinatorics. In these lectures we present some of them. More precisely, we use Alexander duality to study

• combinatorics of simplicial complexes that admit an embedding in low codimension;

• graph theoretic connectivity of manifold graphs;

• Betti numbers of some monomial ideals.

We assume the audience to be familiar with basic concepts of linear algebra and point set topology. A familiarity with simplicial complexes and their homology group would be useful but not necessary, as we recall these notions briefly.

Topics in Topological Combinatorics

Benjamin Matschke
Université de Bordeaux

Topological combinatorics is the study of combinatorial problems via methods from algebraic topology. Combinatorics should be understood here in a broad and usually geometric sense.

Classical topics in this field include Tverberg type theorems, Hom complexes (related to chromatic number of graphs), measure equipartitions, billiards, inscribing and circumscribing problems, Knaster problems, asynchronous computability, Nash equilibria, topological complexity, Euclidean Ramsey theory, the first and second selection lemma, and the Gromov-Milman conjecture.

They rely on basic notions from algebraic topology, which in our setting yield quite often very useful extensions of the classical Borsuk-Ulam theorem: bordisms, obstruction theory, characteristic classes, Fadell-Husseini index, Schwarz genus.

In this lecture series we will give an overview via representative and instructive examples.

Resolutions of letterplace and co-letterplace ideals of poset

Amin Nematbakhsh
IPM

The ideals of poset homomorphisms were introduced and studied by G. Fløystad, B. M. Greve and J. Herzog in 2015 [4]. These are square-free monomial ideals associated with a pair of finite posets P and Q in the polynomial ring $\Bbbk[x_{P\times Q}]$ on indeterminates x_i for $i \in P \times Q$. For any poset homomorphism $\phi : P \to Q$ the graph of ϕ is the set $\Gamma\phi = \{(p, \phi(p))|p \in P\} \subseteq P \times Q$. We can assign to $\Gamma\phi$ a square-free monomial $m_{\Gamma\phi} = \prod_{i\in\Gamma\phi} x_i$ in $\Bbbk[x_{P\times Q}]$. The *ideal of poset homomorphisms* denoted by $L(P, Q)$ is the ideal generated by $m_{\Gamma\phi}$ for all poset homomorphisms $\phi : P \to Q$. In particular, if we set P or Q to be the chain of n elements $[n] = \{1 < \cdots < n\}$ then the ideals $L([n], P)$ and $L(P, [n])$ are called the *letterplace* and *co-letterplace ideals of posets* respectively. These ideals first appeared in the work of V. Ene, J. Herzog and F. Mohammadi [3].

In these series of lectures we investigate the resolutions of letterplace and co-letterplace ideals.

Lecture 1: Letterplace and co-letterplace ideals of posets

In the first lecture we study algebraic properties of these ideals. We show that the letterplace ideal $L(n, P)$ and the co-letterplace ideal $L(P, n)$ are Alexander duals. We also show that the letterplace ideals are Cohen–Macaulay. This implies that the co-letterplace ideals have linear resolutions.

Several homological properties are invariant under dividing out by a regular sequence, most importantly the Betti numbers do not change. It turns out that several classes of ideals in the literature can be derived from these ideals by dividing out with a regular sequence of variable differences (a process called separation). For letterplace ideals these classes include Multichain ideals, initial ideals of determinantal ideals (with respect to a diagonal term order), initial ideals of one-sided ladder determinantal ideals, initial ideal of the ideal of 2-minors of a generic symmetric matrix and initial ideals of certain classes of Pfaffian ideals. Monomial ideals which are regular quotients of co-letterplace ideals include strongly stable ideals generated in a single degree, Ferrers ideals, edge ideals of cointerval d-hypergraphs and uniform face ideals. We study the relation of these ideals to letterplace and co-letterplace ideal as far as time permits.

Lecture 2: Multigraded Betti numbers of letterplace ideals

In the second lecture we provide computational and structural results on multigraded Betti numbers of letterplace ideals. Let $\Delta(n, P)$ be the Stanley-Reisner complex of $L(n, P)$. Given a multidegree $R \subseteq [n] \times P$ we let $\{i\} \times R_i$ be the intersection of R with $\{i\} \times P$ for $i = 1, \ldots, n$. This gives us subsets $R_1, \ldots, R_n$ of P. Each pair (R_i, R_{i+1}) gives rise to a bipartite graph whose edges are $\{p, q\}$ where $p \in \max\{R_i\}$, $q \in \min\{R_{i+1}\}$ and $p \leq q$ in P. Let $X_i(R)$ be the Stanley-Reisner complex of this bipartite graph. We show that the restricted complex $\Delta(n, P)_{|R}$ is homotopy-equivalent to the join $X_1(R) * \cdots * X_{n-1}(R)$.

Furthermore, we study the extremal aspects of the Betti table of $L(n, P)$, like the multidegrees appearing in the first and last linear strands and in the highest homological degree. In the special case that P is a poset for which its Hasse diagram is a rooted tree we describe all the multigraded Betti numbers of $L(n, P)$ and give an inductive procedure to compute the Betti table of such ideals. We show that the Betti table of such letterplace ideals can be computed from letterplace ideals $L(m, Q)$ where $m \leq n$ is an integer and Q is a subposet of P corresponding to a branch in the Hasse diagram of P.

It is worth mentioning that the Betti numbers of letterplace ideals are characteristic dependent and computation of resolutions of letterplace ideals seems to be as difficult as computation of resolutions of Stanley-Resiner rings. More details can be found in [2].

Lecture 3: Resolution of co-letterplace ideals

The set of all poset homomorphisms $\mathrm{Hom}(P, [n])$ is again a poset and one can consider ideals $L(\mathcal{J}) \subseteq L(P, n)$ generated by $m_{\Gamma\phi}$ for $\phi \in \mathcal{J}$, where $\mathcal{J}$ is a poset ideal of $\mathrm{Hom}(P, [n])$. These ideals are called *co-letterplace ideals associated with the poset ideal* $\mathcal{J}$.

The resolutions of co-letterplace ideals $L(P, n)$ are already given in [3] using the fact that these ideals have linear quotients. In the last lecture we construct resolutions of ideals $L(\mathcal{J})$ using a completely different method. Two techniques are commonly used in the literature to construct linear resolutions. First is when the ideal has linear quotients and a regular decomposition function as described in [5]. The second technique is to build a cellular resolution. The technique that we use here is based on the work of Yangawa [6, 7] which uses the canonical module of the Alexander dual to construct the linear resolution explicitly.

In [6], Yanagawa introduced square-free modules over a polynomial ring R, generalizing the notion of square-free monomial ideals. He shows that if M is square-free then all of its syzygy modules and also the modules $\mathrm{Ext}^i_R(M, R)$ are also square-free. For a square-free module M he constructs a chain complex of free R-modules denoted by $L_\bullet(M)$ from the square-free graded components of M and the maps between them. Suppose I is a square-free monomial ideal in R. He shows that for the particular case of $M = \mathrm{Ext}^i_R(R/I, R(-\mathbf{1}))$ the chain complex $L_\bullet(M)$ is isomorphic to the i-linear strand of the resolution of the Alexander dual ideal I^A.

Therefore, if I is a Cohen-Macaulay ideal then the only non-vanishing Ext module is the top one which is the canonical module and the chain complex $L_\bullet(\omega_{R/I})$ is the linear resolution of the Alexander dual ideal I^A. In our case the letterplace ideals are Cohen-Macaulay and we use the canonical module to construct the linear resolutions of ideals $L(\mathcal{J})$. The canonical module of the Alexander dual module of $L(\mathcal{J})$ has the property that it identifies as an ideal in its Stanley-Reisner ring. This is known to happen when $L(\mathcal{J})^A$ is the Stanley-Reisner ideal of a homology ball or a homology sphere. As a consequence we construct a very large class of Gorenstein square-free monomial ideals giving rise to simplicial spheres. In special case when P is an antichain and $n = 2$ we get the class of Bier spheres. For more details see [1].

References

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Tropical Geometry

Johannes Rau
University of Tübingen

Tropical geometry is an exciting new field in mathematics that merges algebraic and symplectic geometry with combinatorics, the result being a fascinating new type of convex geometry. Since its creation tropical geometry has found applications in many different areas of mathematics, from enumerative geometry to phylogenetics and even economics. It my mini-course I want to give a basic introduction to the field, with special emphasis on the beautiful combinatorial aspects. My lectures will be subdivided roughly into the following sections:

1. The Basics: Tropical polynomials, hypersurfaces and varieties.
2. Application 1: Enumerative Geometry in the plane.
3. Application 2: Matroids and Phylogenetics.

Volkmar Welker
Philipps-Universität Marburg

Lecture 1: Ideals of orthogonal embeddings of graphs

An orthogonal embedding of a graph $G = (V, E)$ in $\mathbb{R}^d$ is a map $f : V \to \mathbb{R}^d$ such that for $v, w \in V$, $v \neq w$, we have that $f(v)$ and $f(w)$ are orthogonal if $\{v, w\}$ is not an edge. Considering the coordinates of $\mathbb{R}^d$ as variables the conditions on the embedding define an ideal in d x #V variables generated by quadrics. We provide an overview about the motivation for studying orthogonal embedding and derive algebraic properties of the ideals.

Lecture 2: On the poset of proper divisibility

We define the concept of proper divisibility of monomials and give motivation for studying this concept. In particular, we outline its connection to the Buchberger complex and free resolutions. Finally we show that the poset of proper divisibility is shellable.

Lecture 3: The lattice of subracks of a rack

We define the concept of a rack, which is a set with a binary operation satisfying self-distributivity. For us the prime example is a group with the conjugation as the binary operation. The set of sub-racks ordered by containment is a lattice. We exhibit topological properties of this lattice in the group case and show that group theoretical properties can be read off from the lattice.